NON-ABELIAN FLUX ALGEBRAS IN YANG-MILLS THEORIES *

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Abstract

Contour gauges are discussed in the framework of canonical formalism. We find flux operator algebras with the structure constants of underlying Yang-Mills theory.

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1 Introduction

Contour gauges [1] have been already successfully applied to Yang-Mills theories in the framework of path integral approach. In this note I am going to discuss these gauges in the framework of canonical formalism. The curves admitted will be those discussed in ref.[2], where a slight modification of the gauge condition used in [1] has been proposed and discussed under the name of ponderomotive gauges. These gauges can be indexed [2] with homotopy families admissible by geometry of region V considered. In the case of Yang-Mills theory we limit ourselves to self-contractible families, defined in ref.[2]. They have a useful property, namely Y-M potentials are orthogonal to these curves, i.e. from the gauge constraints

$$\int_{c(x,x_0)} A_a^\mu(y) dy_\mu = 0 \tag{1.1}$$

follows [2]

$$\frac{\partial c_\mu(x,x_0,\tau)}{\partial \tau} A_a^\mu(c(x,x_0,\tau)) = 0 \tag{1.2}$$

for any $0 \leq \tau \leq 1$ and $x \in V$.
We are going to implement these gauges into canonical formalism of Y-M theory (Ch.2). Next, we establish in Ch.3 algebras of fluxes:

$$\mathcal{B}_a^{(\sigma)} = \int_\sigma B_a^k n^k d\sigma \tag{1.3}$$

$$\mathcal{E}_a^{(\sigma^*)} = \int_{\sigma^*} E_a^k n^k d\sigma \tag{1.4}$$

$$\vec{x}^0 \notin \sigma$$

2 Dirac brackets for Y-M theory

In what follows the discussion of surface terms will be omitted. The canonical Hamiltonian is then:

$$H = \int_V d^3x \mathcal{H} \tag{2.1}$$

with

$$\mathcal{H} = \frac{1}{2}(\vec{B}_a \cdot \vec{B}_a + \vec{E}_a \cdot \vec{E}_a) - [\vec{\nabla} \cdot \vec{E}_a - g\mathcal{C}_{abc}\vec{A}_b \cdot \vec{E}_c] A_a^0 \tag{2.2}$$

$$E^i_a(\vec{x}) = -\Pi^i_a(\vec{x}) \tag{2.3}$$

$$D^{(1)}_a = \Pi^0_a \approx 0 \tag{2.4}$$

$$D^{(2)}_a = \vec{\nabla}\cdot\vec{E}_a - g\mathcal{C}_{abc}\vec{A}_b\cdot\vec{E}_c \approx 0 \tag{2.5}$$

We take temporal gauge

$$D^{(3)}_a = A^0_a \approx 0 \tag{2.6}$$

and ponderomotive space-like gauge constraint

$$D^{(4)}_a = \int_{c(\vec{x},\vec{x}_0)} A^i_a(y)dy^i \approx 0 \tag{2.7}$$

The constraints (2.4-2.7) are compatible with

$$\mathcal{H}' = \frac{1}{2}(\vec{B}_a\cdot\vec{B}_a + \vec{E}_a\cdot\vec{E}_a) + D^{(2)}_a(x)v^{(2)}_{(a)}(x) \tag{2.8}$$

where

$$v^{(2)}_a(x) = \int_{c(\vec{x},\vec{x}_0)} dy^i E^i_a(y) \tag{2.9}$$

Let us notice that the condition (2.7) is trivial for $\vec{x} = \vec{x}_0$ (for self-contractible curves $\vec{c}(\vec{x}_0,\vec{x}_0) = \vec{x}_0$). Therefore our further considerations will be valid for the region V_-:

$$V_- = V - P(\vec{x}_0) \tag{2.10}$$

Next, let us remark that compatibility of (2.7) with (2.8) is evident once we prove - in analogy with Maxwell theory [2]- that in V_-

$$[D^{(4)}_d(\vec{x}), D^{(2)}_a(\vec{y})]_P = \delta(\vec{x}-\vec{y})\delta_{ad} \tag{2.11}$$

The first term of $D^{(2)}_a(y)$, $-\vec{\nabla}\cdot\vec{E}_a$ (comp. eq.(2.5))yields already r.h.s. of (2.11) - derivation is the same as for Maxwell theory (comp.[2]). So we have to show that

$$\mathcal{C}_{abc}[D^{(4)}_a(\vec{x}), A^i_b E^i_c(y)]_P \approx 0 \tag{2.12}$$

The use of (2.7) gives

$$\begin{aligned} &\mathcal{C}_{abc}[D^{(4)}_a(x), A^i_b(y)E^i_c(y)]_P = \\ &= \mathcal{C}_{abc}\int_{c(\vec{x},\vec{x}_0)} dz^k [A^k_d(z), A^i_b(y)E^i_c(y)]_P = \\ &= \mathcal{C}_{abc}(-)\delta_{cd}\int_{c(\vec{x},\vec{x}_0)} dz^k \delta(\vec{z}-\vec{y})A^k_b(z) \end{aligned} \tag{2.13}$$

Please notice, that $dz^k A^k(z)|_{z\in c(\vec{x},\vec{x}_0)} \approx 0$ from (2.7) (comp.eqns (1.1), (1.2)), therefore (2.12) is proved.
Let us come back to constraints ((2.4)-(2.7)). With the help of (2.11)the matrix

$$d^{i,k}_{a,b} = [D^{(i)}_a(x), D^{(k)}_b(y)]_P \tag{2.14}$$

can be written as

$$d^{i,k}_{a,b} = \begin{bmatrix} 0 & -I \\ I & 0 \end{bmatrix}^{ik} \cdot \delta_{ab}\delta_3(\vec{x}-\vec{y}) \tag{2.15}$$

so that

$$d^{-1} = -d \tag{2.16}$$

and Dirac brackets of the theory follow [3]:

$$[E^r_c(\vec{x}), A^s_d(\vec{y})]_D = \delta_{cd}\delta_{rs}\delta(\vec{x}-\vec{y}) - \\ - \left[\frac{\partial}{\partial y^s}\delta_{cd} - g\mathcal{C}_{cdb}A^s_b(y)\right] \cdot \int_{w\in c(\vec{y},\vec{x}_0)} dw^r \delta(\vec{x}-\vec{w}) \tag{2.17}$$

$$[A^r_c, A^s_d]_D = 0 \tag{2.18}$$

$$[E^r_c(\vec{x}), E^s_d(\vec{y})]_D = \\ = g\mathcal{C}_{cdf}\left[\int_{w\in c(\vec{x},\vec{x}_0)} dw^s \delta(\vec{y}-\vec{w})E^r_f(\vec{x}) + \int_{w\in c(\vec{y},\vec{x}_0)} dw^r \delta(\vec{x}-\vec{w})E^s_f(y)\right] \tag{2.19}$$

In the next section eqns.(2.17 - 2.19) will be used in the derivation of non-abelian algebras.

3 Flux algebras

Let us consider at the beginning a special type of surfaces appearing in definitions (1.3), (1.4) of $\mathcal{B}$, $\mathcal{E}$ fluxes. Take a loop L and some homotopy $c(\vec{x},\vec{a})$. We define a horn $H(L,c)$:

$$\vec{x} \in H(L,c) \Longleftrightarrow x^k = c^k(\vec{L}(t), \vec{a}, t_1) \tag{3.1}$$

for some $t, t_1 \in [0,1]$ and fix orientation on this surface:

$$\vec{n} || \left(\frac{\partial \vec{c}}{\partial t_1} \times \frac{\partial \vec{c}}{\partial t} \right) \tag{3.2}$$

We are going to show that fluxes $\mathcal{B}, \mathcal{E}$ through these homotopy horns are equal to loop integrals:

$$\int_{H(L,c)} (\vec{B}_a \cdot \vec{n}) d\sigma = \int_L f_a^r dx^r \tag{3.3}$$

$$\int_{H(L^*,c^*)} (\vec{E}_a \cdot \vec{n}) d\sigma = \int_L .^* f_a^r dx^r \tag{3.4}$$

with

$$f_a^r(x) = \int_c B_a^k \varepsilon^{kij} \frac{\partial y^j}{\partial x^r} dy^i \tag{3.5}$$

$$^* f_a^r(x) = \int_{c^*} E_a^k \varepsilon^{kij} \frac{\partial y^j}{\partial x^r} dy^i \tag{3.6}$$

L, L^* and c, c^* need not be related. At this stage we need not specify in what gauge $\vec{B}_a, \vec{E}_a$ are given. Eqns.(3.3), (3.4) are consequence of a simple observation. Take any antisymmetric tensor T_{ij} and define

$$g^r(x) = \int_{y \in c(\vec{x}, \vec{a})} T^{ij}(y) \frac{\partial y^j}{\partial x^r} dy^i \tag{3.7}$$

Then

$$\int_L g^r(x) dx^r = \int dt dt_1 \frac{\partial y^i}{\partial t_1} \frac{\partial y^j}{\partial t} \varepsilon^{ijk} T^k(y) \tag{3.8}$$

where

$$T^{ij} = \varepsilon^{ijk} T^k \tag{3.9}$$

$$y^i = c^i(L(t), x_0, t_1) \tag{3.10}$$

$$\int_L g^r(x) dx^r = \int_{H(L,c)} (\vec{T} \cdot \vec{n}) d\sigma \tag{3.11}$$

Replacement $g \longrightarrow for^* f$ and $T^k \longrightarrow B^k or E^k$, gives eqns.(3.3) and (3.4), respectively.
If we specify $\vec{B}_a$, $\vec{E}_a$ to be in a gauge defined through c from eqn.(3.5), then $f_a^r(x)$ is a potential in this gauge (comp.ref.[2]). Still there is a vast choice

of homotopies c^* defining dual potential $,^* f$. Let us consider Dirac brackets of $\mathcal{E}, B$ in c-gauge:

$$[\mathcal{B}_d, \mathcal{E}_c]_D = \left[\int_L f_d^r dx^r, \int_{L^*} .^* f_c^s dy^s\right]_D =$$
$$= \left[\int_L f_d^r dx^r, \int_{L^*} dy^s \int_{c^*(y,a^*)} dz^i \frac{\partial z^j}{\partial y^s} \varepsilon^{ijk} E_c^k(z)\right]_D \quad (3.12)$$

Using (2.17) one gets, after some algebra, the following expression:

$$\left[\mathcal{E}_c^{H^*}, \mathcal{B}_d^H\right]_D = \delta_{cd} N(L; H^*) +$$
$$+g c_{cdg} \int_{x \in L} dx^r f_g^r(x) N(c(x,a); H^*) \quad (3.13)$$

where

$$N(L; H^*) = \sum_t sgn\left(\frac{\partial \vec{L}(t_1)}{\partial t_1} \cdot \vec{n}_{H^*}(t_2, t_3)\right) \quad (3.14)$$

$$N(c(\vec{x}, a); H^*) = \sum_{\tau(x)} sgn\left(\frac{\partial \vec{c}(\vec{x}, \vec{a}, \tau_1)}{\partial \tau_1} \cdot \vec{n}_{H^*}(\tau_2, \tau_3)\right) \quad (3.15)$$

with t_i, $\tau_i(x)$ being the solutions of the following equations:

$$\vec{c}^*(L^*(t_2), \vec{a}^*, t_3) = \vec{L}(t_1) \quad (3.16)$$

$$\vec{c}^*(L^*(\tau_2), \vec{a}^*, \tau_3) = \vec{c}(\vec{x}, \vec{a}, \tau_1) \quad (3.17)$$

and $\vec{n}_{H^*}$ being normal to a horn $H^* \equiv H(L^*, c^*)$ (comp.eqns (3.1), (3.2)). The conditions (3.16) or (3.17) are fulfilled whenever the surface of H^* is pierced by loop L or homotopy curve $c(\vec{x}, \vec{a})$, respectively. N's in eqns (3.14), (3.15) denote net numbers of piercings.

The abelian part of (3.13) has been already discussed [4] for the radial gauge; it leads to t'Hooft algebra [5]. The non-abelian part can be expressed through surface integrals. Call K_N part of a loop L, characterized by $N(c; H^*) = N$, N fixed (K_N can consist of disjoint pieces). We have $L = \sum_N K_N$ and corresponding horn surface:

$$H(L, c) = \bigcup_N H(K_N, c) \quad (3.18)$$

where

$$x \in H(K_N, c) \Longleftrightarrow \vec{x} = \vec{c}(\vec{y}, \vec{a}, t) \quad (3.19)$$

for some $Y \in K_N$ and $t \in [0,1]$. Evidently eqn(3.3) holds for $H(K_N, c)$ so that eqn(3.13) can be rewritten as:

$$[\mathcal{E}_c^{H^*}, \mathcal{B}_d^H] = \delta_{cd} N(L; H^*) + \\ + g c_{cdg} \sum_N N \int_{H(K_N,c)} \vec{B}_g \cdot \vec{n} d\sigma \tag{3.20}$$

Let us add, that in fact eqn(3.20) holds for any surface S, not necessarily a horn $H^*(L^*, c^*)$. H^* is useful if we want to keep relation with loop integrals over dual potential (see eqns (3.4), (3.6)). More generally, we have:

$$[\mathcal{E}_c^S, \mathcal{B}_d^H] = \delta_{cd} N(L; S) + g c_{cdf} \sum_N N \int_{H(K_N,c)} \vec{\mathcal{B}}_f \cdot \vec{n} d\sigma \tag{3.21}$$

Let us consider now fluxes $\mathcal{E}_c^{(S_1)}$, $\mathcal{E}_c^{(S_2)}$. Surfaces S_i are parametrized by given $s_i(t_1, t_2)$:

$$x \in S_i \Longleftrightarrow \vec{x} = \vec{s}_i(t_1, t_2) \tag{3.22}$$

for some $(t_1, t_2) \in [0,1]$.

The Dirac bracket of $\mathcal{E}_c^{(S_1)}$, $\mathcal{E}_d^{(S_2)}$ - calculated in c-gauge - is given by the following expression:

$$[\mathcal{E}_c^{(S_1)}, \mathcal{E}_d^{(S_2)}] = \\ g C_{cdf} \left[\int_{s_1 \in S_1} \vec{E}_f(s_1) \vec{n}_{S_1}(s_1) N(c(\vec{s}_1, \vec{a}); S_2) d\sigma + \right. \\ \left. + \int_{s_2 \in S_2} \vec{E}_f(s_2) \vec{n}_{S_2}(s_2) N(c(\vec{s}_2, \vec{a}); S_1) d\sigma \right] \tag{3.23}$$

where N's are the net numbers of piercings:

$$N(c(\vec{s}_1, \vec{a}); S_2) = \sum_{t_i(s_1)} sgn \left(\frac{\partial \vec{c}(\vec{s}_1, \vec{a}, t_3)}{\partial t_3} \cdot \vec{n}_{S_2}(t_1, t_2) \right) \tag{3.24}$$

with $t_i(s_1)$ being solutions of the following equation:

$$\vec{c}(\vec{s}_1, \vec{a}, t_3) = \vec{s}_2(t_1, t_2) \tag{3.25}$$

Eqn (3.25) is fulfilled whenever, for a given $s_1 \in S_1$ the homotopy curve $c(\vec{s}_1, \vec{a})$ crosses the surface S_2. Changing $s_1 \to s_2$, $S_1 \to S_2$ in eqns (3.24),

(3.25) one gets N from the second integral on the r.h.s. of eqn(3.23). Making in (3.23) transition $S_2 \to S_1$ we get for $S_1 = S_2 = S$:

$$[\mathcal{E}_c^{(S)}, \mathcal{E}_d^{(S)}]_D = 2gc_{cdf} \int_{s \in S} \vec{E}_f(s) \cdot \vec{n}_S(s) N(c(s,a); S) d\sigma \tag{3.26}$$

In this case there is always at least one common point of $c(\vec{s}, \vec{a})$ and S, as $c(\vec{s}, \vec{a})$ ends on $s \in S$. The weight of this end- point contribution to N is $\frac{1}{2}$ as can be seen from the limiting transition $S_1 \to S_2$ in eqn(3.23). Therefore, for any fixed $s \in S$:

$$\begin{aligned} 2N(c(s,a); S) &= sgn \frac{\partial \vec{c}(\vec{s}, \vec{a}, t_3)}{\partial t_3}|_{t_3=1} \cdot \vec{n}_S(\vec{s}) + \\ +2 \sum_{t; t_3 \neq 1, s \neq s(t_1,t_2)} & sgn \left(\frac{\partial c(\vec{s}, \vec{a}, t_3)}{\partial t_3} \cdot \vec{n}_S(t_1, t_2) \right) \end{aligned} \tag{3.27}$$

with

$$\vec{c}(\vec{s}, \vec{a}, t_3) = \vec{s}(t_1, t_2) \tag{3.28}$$

Eqns (3.21), (3.26) together with the trivial bracket

$$[\mathcal{B}_c, \mathcal{B}_d]_D = 0 \tag{3.29}$$

do not form closed algebra for any chosen $H(L, c)$ and S. They can be however replaced by a set of closed algebras on the properly chosen parts of $H(L, c)$ and S. This will be discussed elsewhere [6]. Let us conclude with a choice of such $H(L_0, c)$ and S_0 that

$$L_0 \bigcup S_0 = \emptyset \tag{3.30}$$

i.e. abelian part does not contribute to (3.21). Moreover, put $2N$ in eqn(3.26) and N in eqn(3.21) equal to 1. (Example: in the Fock-Schwinger gauge take $H(L_0, C)$ to be a cone and S_0 to be any planar surface containing elliptic section of H_0). In such a case we have:

$$[\mathcal{E}_a^{S_0}, \mathcal{B}_b^{H_0}]_D = gc_{abc} \mathcal{B}_c^{H_0} \tag{3.31}$$

$$[\mathcal{E}_a^{S_0}, \mathcal{E}_b^{S_0}]_D = gc_{abc} \mathcal{E}_c^{S_0} \tag{3.32}$$

$$[\mathcal{B}_a^{H_0}, \mathcal{B}_b^{H_0}]_D = 0 \tag{3.33}$$

If we took S^0 to be closed surface surrounding $\vec{a}$, then (3.32) is the algebra of charges contained in its interior, V_0:

$$[Q_a^E(V_0), Q_b^E(V_0)]_D = gc_{abc} Q_c^E(V_0) \tag{3.34}$$

The question whether replacement of fluxes $\mathcal{B}^H$ by $\mathcal{B}^\sigma$ - σ being arbitrary surface - leaves simplicity of flux algebras intact will be discussed elsewhere [6].

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